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A modification of the Wiener process due to a Poisson random train of diffusion-enhancing pulses

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Abstract. A Poisson-modified Wiener process is considered. Its conditional probability density is calculated exactly. Various forms of the evolution equation are derived for the case when the initial probability density is arbitrary. A generalization is also treated when this equation contains a term analogous to the potential energy term in the Schrödinger equation. The Green function of this equation is derived in the form of a functional integral which may be considered as a direct generalization of the Feynman-Kac integral. An application is suggested in the theory of quasiparticles with a non-parabolic dispersion law.

1. Introduction

In the present paper, we shall describe a multiplicative stochastic process combining two processes: the Wiener process (i.e. classical Brownian motion) with a diffusion constant D and a Poisson random train of equal, very short-lasting, but highly diffusion-enhancing, pulses. The duration τ_p and the diffusion coefficient D_p of each pulse, respectively, are assumed to tend to zero and infinity. The product

$$D_p\tau_p = M \tag{1}$$

for $\tau_p \to 0$, $D_p \to \infty$, however, is kept constant. We call M the strength of the pulses. For brevity, we shall speak of M-pulses. The second parameter defining the Poisson train of the pulses is their average frequency ν. Thus, our multiplicative process is defined altogether by three positive constants: D, M and ν. The problem dealt with in this paper has been suggested by Laskin (Laskin 1989). Our problem is a generalization of Laskin's: our process is reduced to his if $D \to +0$.

Laskin's motivation to put forward a theory of non-Gaussian diffusion was related to his theory of channelled relativistic electrons in crystals. In one of his two models, he attributed the dechannelling of the electrons to short, but active, fluctuation regions forming a Poisson random-layered structure in any channel. Each layer was taken with the same thickness, the value of which was small in comparison with the average distance between the layers. Then an electron travelling with some velocity across the crystal (along a channel) is subjected to shocks (pulses), the distribution of which in time is Poissonian. That is why Laskin's 'spatial terminology' (the Poisson layering) is in conformity with the 'temporal terminology' (the Poisson train of the pulses) which we prefer.

The scope of the present paper is as follows. In section 2 we recall some basic results of the classical theory of Brownian motion. Afterwards, in section 3, we define the main topic which we are concerned with in this paper—the Poisson-modified

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Wiener process $\xi(u)$—by using a diagrammatic method. This method is analogous to the summation over Feynman's diagrams that is so well known in quantum theory. For the process $\xi(u)$, we derive the conditional probability density $P(x, t|x_0)$. In section 4 we calculate all the cumulants of the random variable x, thus showing explicitly that the process $\xi(u)$ is non-Gaussian. It is, however, Markovian (section 5). In section 6 we derive three formally different, but essentially equivalent, versions of the evolution equation for a general probability density $\psi(x, t)$ concerning some ensemble of simultaneously running processes $\xi(u)$ (in the sense that each process $\xi(u)$ may start from its own point of departure so that the initial probability density $\psi(x, 0) = \psi_0(x)$ need not be the delta function centred at some fixed position x_0). In sections 7 and 8 we present a functional integral form for $P(x, t|x_0)$ (i.e. in fact, for the Green function of the evolution equation(s) derived in section 6). In particular, we generalize the equation(s) by considering, in section 8, a term resembling the potential energy term in the Schrödinger equation. So, $P(x, t|x_0)$ becomes a functional of a 'potential' $V(x)$. Our functional integral representation of $P(x, t|x_0)$ (actually an infinite product of integrals—a 'product integral') generalizes the Feynman–Kac integral known from the classical theory of Brownian motion and from quantum mechanics. In section 9, we propose an application of our process $\xi(u)$: we can model the dynamics of quasiparticles if their dispersion law (i.e. the dependence of their energy on their wavevector) exhibits a certain kind of deviation from parabolicity. Finally, section 10 is a summary.

2. Some preliminaries concerning the classical diffusion process

Let $G_0(x, t|x_0)$ be the probability density of a Brownian particle starting at $t_0 = 0$ from position x_0. (We will confine ourselves to one-dimensional problems in the present paper.) We assume that $t \geq 0$ and $-\infty < x < \infty$. As was shown by Einstein in 1905 (e.g. cf Wax 1954), G_0 obeys the diffusion equation

$$\partial G_0(x, t|x_0)/\partial t = D\partial^2 G_0(x, t|x_0)/\partial x^2 \tag{2}$$

and the initial condition

$$G_0(x, +0|x_0) = \delta(x - x_0). \tag{3}$$

Obviously,

$$\int_{-\infty}^{\infty} \mathrm{d}x\, G_0(x, t|x_0) = 1. \tag{4}$$

If the particle moves in a medium where it may become immobilized or annihilated with a rate $V(x) \geq 0$, we may write a generalized equation:

$$\partial G(x, t|x_0)/\partial t = D\partial G(x, t|x_0)/\partial x^2 - V(x)G(x, t|x_0). \tag{5}$$

(A certain interpretation of equation (5) can be invented even if $V(x) < 0$: we may imagine a generation of diffusive particles in the medium.) If $V(x)$ does not vanish everywhere, condition (4) need not be satisfied. The fundamental solution to equation (5) can be written in the form of the Feynman–Kac functional integral (path integral):

$$G(x, t|x_0) = \mathcal{N}(t) \int_{x_0,0}^{x,t} D\xi(u) \exp\left(-\frac{1}{4D}\int_0^t \mathrm{d}u\, \dot{\xi}^2(u) - \int_0^t \mathrm{d}u\, V(\xi(u))\right) \tag{6}$$

$$G(x, +0|x_0) = \delta(x - x_0). \tag{7}$$

Here we have used the notation customary in Feynman's formulation of quantum mechanics (Feynman and Hibbs 1965, Papadopoulos and Devreese 1977, Schulman 1984). In the mathematical theory of stochastic processes, this notation may be used (Kac 1967) if one abstracts from such fine notions as the Wiener measure (Yeh 1973, Hida 1980). The factor $\mathcal{N}(t)$ is a (positive) normalizing coefficient and $\xi(u)$, for $0 \leqslant u \leqslant t$, is an arbitrary path connecting the end-points $\xi(0) = x_0$ and $\xi(t) = x$. If the time is discretized, then—using $\varepsilon = t/N$, $u_j = j\varepsilon$, $\xi_j = \xi(u_j)$—we may write the multiple integral

$$G(x, t|x_0) = \frac{1}{(4\pi D\varepsilon)^{N/2}} \left(\prod_{j=1}^{N-1} \int_{-\infty}^{\infty} d\xi_j \right) \exp\left(-\frac{1}{4\pi\varepsilon} \sum_{i=0}^{N-1} (\xi_{i+1} - \xi_i)^2 - \varepsilon \sum_{i=0}^{N-1} V(\xi_i) \right) \quad (8)$$

provided that $N \to \infty$ and $\xi_0 = x_0$, $\xi_N = x$. Thus, $\dot{\xi}(u_i)$ is understood as $(\xi_{i+1} - \xi_i)/\varepsilon$. (In the Wiener theory, the terms with $\dot{\xi}(u_i)$ are incorporated in the Wiener measure.) Note that all the integrals $\int du$ correspond to the Itô definition of the stochastic integrals (e.g. cf Ikeda and Watanabe 1981). In our further formulations in this paper, there will be no necessity to distinguish between the Itô and Stratonovich integrals $\int du$ (e.g. cf Gardiner 1986). When employing the N-approximation, we may rewrite G in the form of the product integral

$$G(x, t|x_0) = \left(\prod_{j=0}^{N-1} \int_{-\infty}^{\infty} d\xi_j \, G_0(\xi_{j+1}, \varepsilon|\xi_j) \exp(-\varepsilon V(\xi_j)) \right) \delta(\xi_0 - x_0) \quad (9)$$

(with $\xi_N = x$) where

$$G_0(\xi_{j+1}, \varepsilon|\xi_j) = \frac{1}{(4\pi D\varepsilon)^{1/2}} \exp\left(-\frac{(\xi_{j+1} - \xi_j)^2}{4D\varepsilon} \right). \quad (10)$$

In section 8, we shall derive a generalization of integral (9) showing how the parameter M enters the functional integral theory devised for our multiplicative process $\xi(u)$.

3. A diagrammatic method of defining the Poisson-modified Wiener process

Let us introduce a time-varying diffusion coefficient $D(t)$. Then we may write the equation

$$\partial G(x, t|x_0)/\partial t = D(t)\partial^2 G(x, t|x_0)/\partial x^2 \quad (11)$$

$$G(x, 0|x_0) = \delta(x - x_0). \quad (12)$$

(We take $V(x) \equiv 0$ for simplicity.) Now let us define $D(t)$ as a two-state stochastic process. This means that $D(t)$ may acquire two constant values which we denote as D and D_p, $0 \leqslant D < D_p$. So let two Wiener processes be alternating: one with the coefficient D, one with the coefficient D_p. For brevity, we say that the Brownian particle may exist either in the D-state or in the D_p state. Let t_i^D be any time instant when the particle begins to dwell in the D-state and t_f^D the next time instant when the particle leaves this D-state, going over into the D_p state. We define $t_f^D - t_i^D$ as a Poisson random variable so that the average duration of the D-state is

$$\langle t_f^D - t_i^D \rangle = \nu \int_0^{\infty} d\tau \, \tau \exp(-\nu\tau) = \nu^{-1}. \quad (13)$$

(We shall use the brackets $\langle\ \rangle$ to denote averaging with respect to the Poisson randomness of $D(t)$.) The duration of the D_p state is kept constant.

The formal solution of equation (11) is

$$G(x, t\,|\,x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \exp[\mathrm{i}k(x-x_0)]\, \exp\left(-k^2 \int_0^t \mathrm{d}\tau\, D(\tau)\right). \tag{14}$$

The function $G(x, t\,|\,x_0)$ is stochastic, owing to $D(\tau)$. Nevertheless, the probability density meaning is maintained:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, G(x, t\,|\,x_0) = 1. \tag{15}$$

It is, therefore, natural to deal with the averaged density function

$$P(x, t\,|\,x_0) = \langle G(x, t\,|\,x_0)\rangle. \tag{16}$$

We might also define more expressively the probability densities $P_{\mathrm{fi}}(x, t\,|\,x_0)$ by a conditional averaging:

$$P_{\mathrm{fi}}(x, t\,|\,x_0) = \langle G(x, t\,|\,x_0)\rangle_{\mathrm{i}\to\mathrm{f}}. \tag{17}$$

The indices i, f correspond here to two (possibly identical) states: S_i and S_f—the initial state at time $t_0 = 0$ and the final state at the time $t > 0$. Thus, in general, if ν, τ_p are both non-zero and finite, there are four possibilities for the 'transitions' $\mathrm{i}\to\mathrm{f}$: $D\to D$, $D\to D_p$, $D_p\to D$, $D_p\to D_p$. Then we expect fulfilment of the condition

$$\sum_f \int_{-\infty}^{\infty} \mathrm{d}x\, P_{\mathrm{fi}}(x, t\,|\,x_0) = 1 \tag{18}$$

where the sum runs over the two states D and D_p, whilst the fixed index i may be either of these two states. However, as we take into account the limiting case when $\tau_p\to+0$, our further analysis is reduced to the sole relevant transition: $D\to D$. In this situation, we identify P_{DD} with P.

That is why, when discussing the probability density (16), we may say that $P(x, t\,|\,x_0)$ is given by the sum over the graphs which are depicted in figure 1. According to formula (14), the graphs in figure 1 correspond to the function

$$p_{DD}(k, t) = \left\langle \exp\left(-k^2 \int_0^t \mathrm{d}\tau\, D(\tau)\right)\right\rangle_{DD}. \tag{19}$$

The horizontal direction in each graph corresponds to the time variable τ running from $\tau_i = 0$ to $\tau_f = t$. (That is why each graph is depicted with the same horizontal length.) The arrows symbolize a propagation: the lower arrow in the D-state, the upper one in the D_p state. When all such graphs are exhausted (with all possible 'interstates'), then—by summing up the contributions due to all the graphs—we can obtain the function $p_{DD}(k, t)$ exactly. The lengths of the lower arrows (in the graphs with the 'interstates', i.e. except in the first graph) are random and the upper arrows have the constant length τ_p. After the limiting procedure (1), the upper arrows cease to exist

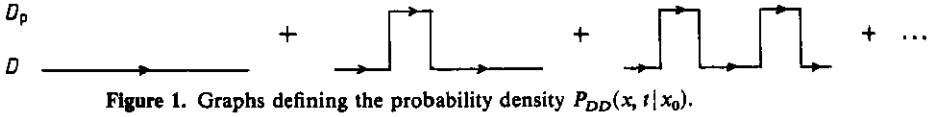

Figure 1. Graphs defining the probability density $P_{DD}(x, t\,|\,x_0)$.

but we must duly take into consideration the M-pulses. Each M-pulse puts the same factor $\nu \exp(-k^2 M)$ in each term of the function $p(k, t)$ calculated according to formula (19). (From now on, we omit the indices DD.) If a graph involves n M-pulses, their total contribution in the corresponding term of $p(k, t)$ is $\nu^n \exp(-nk^2 M)$. This is our first rule. The second rule of our diagrammatic method is the definition of 'propagators' between two M-pulses: we confer the function $\exp[-\nu(\tau_f' - \tau_i')] \exp[-k^2 D(\tau_f' - \tau_i')]$ to each arrow (in our case, some lower arrow only) starting at time τ_i' and ending at time τ_f'. Indeed, the expression $\nu \exp(-\nu\tau)\, d\tau$ equals the probability—according to the Poisson statistics—that there is no M-pulse within the interval $(0, \tau)$ and afterwards, within the interval $(\tau, \tau + d\tau)$, there is just one M-pulse ending the former D-state. As the time instants of the M-pulses are random, our third rule says that we must integrate with respect to all the instants. In this way, we obtain the series

$$\begin{aligned} p(k, t) = {} & \exp(-\nu t) \exp(-k^2 D t) + \nu \int_0^t d\tau_1 \exp(-\nu\tau_1) \exp(-k^2 D\tau_1) \exp(-k^2 M) \\ & \times \exp[-\nu(t-\tau_1)] \exp[-k^2 D(t-\tau_1)] \\ & + \nu^2 \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \exp(-\nu\tau_1) \exp(-k^2 D\tau_1) \exp(-k^2 M) \\ & \times \exp[-\nu(\tau_2-\tau_1)] \exp[-k^2 D(\tau_2-\tau_1)] \\ & \times \exp(-k^2 M) \exp[-\nu(t-\tau_2)] \exp[-k^2 D(t-\tau_2)] + \dots . \end{aligned}$$

So that we can write

$$\begin{aligned} p(k, t) &= \left\langle \exp\left[-k^2 \int_0^t d\tau\, D(\tau) \right] \right\rangle_{\tau_p \to 0, D \to \infty, \tau_p D \to M} \\ &= \exp[\![-\{\nu[1 - \exp(-k^2 M)] + k^2 D\} t]\!]. \end{aligned} \tag{20}$$

When this function is inserted into formula (14) and the integration with respect to k is carried out, we obtain the result

$$P(x, t | x_0) = \frac{1}{(4\pi)^{1/2}} \exp(-\nu t) \sum_{n=0}^{\infty} \frac{(\nu t)^n}{(Dt + nM)^{1/2}} \frac{1}{n!} \exp\left(-\frac{(x - x_0)^2}{4(Dt + nM)} \right). \tag{21}$$

Evidently, if $M \to 0$, then we obtain the simple Gaussian of the classical theory of diffusion:

$$\lim_{M \to 0} P(x, t | x_0) = \Gamma(x - x_0, 2Dt). \tag{22}$$

We have used

$$\Gamma(x, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{x^2}{2\sigma^2} \right) \tag{23}$$

here. Each term in series (21) is positive. For all values of $M \geqslant 0$ (as well as for all values of x_0 and $t \geqslant 0$), we can directly check the identity

$$\int_{-\infty}^{\infty} dx\, P(x, t | x_0) = 1 \tag{24}$$

so that P, as we expected, is undoubtedly a probability density. For $M > 0$, formula (21) shows that P is a linear combination of an infinite set of Gaussians

$$P(x, t|x_0) = \exp(-\nu t) \sum_{n=0}^{\infty} \frac{(\nu t)^n}{n!} \Gamma(x - x_0, 2(Dt + nM)). \tag{25}$$

However, the linear combination of different Gaussians is no longer a Gaussian. This proves that $P(x, t|x_0)$ describes (with the exception of the limiting case when $M = 0$) a genuine non-Gaussian process.

4. Cumulants of the end-point x

When attention is focused on the random paths $\xi(u)$ $(0 \leqslant u \leqslant t)$, then the most natural idea is to theorize using the stochastic differential equation describing them. For our process, this equation reads

$$\dot{\xi}(u) = (2D(u))^{1/2} f(u) \tag{26}$$

where

$$\xi(0) = x_0 \tag{27}$$

is fixed. The RHS of equation (26) involves the standard Gaussian white-noise function $f(u)$ (centred to zero) and the Poisson-type process $D(u)$ defined above. (Clearly, $D(u)^{1/2}$ is a process of the same type as $D(u)$, with two values, $D^{1/2}$ and $D_p^{1/2}$, before the limiting procedure (1) is applied.) We may interpret $\xi(u)$ as a multiplicative process since when we have to average any functional $F\{\xi(u)\}$, we must meticulously say whether we have in mind the averaging $\langle\ \rangle_G$ with respect to the Gaussian process $f(u)$, which is defined by the conditions

$$\langle f(u)\rangle_G = 0 \qquad \langle f(u_1) f(u_2)\rangle_G = \delta(u_1 - u_2) \tag{28}$$

or the averaging $\langle\ \rangle_P$ with respect to the Poisson process $D(u)$ defined by its parameters M and ν. Or we may have in mind the total averaging:

$$\langle\ \rangle = \langle\langle\ \rangle_P\rangle_G = \langle\langle\ \rangle_G\rangle_P. \tag{29}$$

With these definitions, we can identify the functions G, P (cf formulae (14) and (16)) as the following average values:

$$G(x, t|x_0) = \langle \delta(x - \xi(t))\rangle_G \tag{30}$$

$$P(x, t|x_0) = \langle G(x, t|x_0)\rangle_P = \langle \delta(x - \xi(t))\rangle. \tag{31}$$

Obviously, if the time $t > 0$ is kept constant, the end-position $\xi(t) = x$ is a single random variable. For each realization of $D(u)$ (i.e. if we choose any deterministic positive function for $D(u)$), the process is still Gaussian. The non-Gaussian nature arises, of course, from the Poisson statistics of $D(u)$.

We will now calculate the cumulants of the random variable x. Because of the translational invariance of the probability density $P(x, t|x_0)$, we may take $x_0 = 0$. To define the cumulants, we introduce the characteristic function

$$X(\alpha, t) = \langle \exp(\alpha x)\rangle = \int_{-\infty}^{\infty} dx \exp(\alpha x) P(x, t|0). \tag{32}$$

Let us take

$$X(-ik, t) = p(k, t). \tag{33}$$

Indeed,

$$P(x, t|0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp(ikx)p(k, t). \tag{34}$$

The nth cumulant c_n is given by the expression

$$c_n(t) = \frac{\partial^n \ln X(\alpha, t)}{\partial \alpha^n}\bigg|_{\alpha=0} = i^n \frac{\partial^n \ln p(k, t)}{\partial k^n}\bigg|_{k=0}. \tag{35}$$

From expression (20), we see that

$$\ln p(k, t) = -\{\nu[1-\exp(-k^2M)]+k^2D\}t. \tag{36}$$

As this function is even in k, all cumulants with an odd index $2n-1$ ($n = 1, 2, \ldots$) are equal to zero. For the first even index we have the result

$$c_2(t) = 2(\nu M + D)t. \tag{37}$$

For the remaining cumulants c_{2n} with $n > 1$, we have the formula

$$c_{2n}(t) = (-1)^n \nu t M^n \frac{d^{2n}\exp(-\mu^2)}{d\mu^{2n}}\bigg|_{\mu=0}.$$

(We have taken $Mk^2 = \mu^2$.) On taking into account the definition of the Hermite polynomials,

$$H_{2n}(\mu) = \exp(\mu^2)\frac{d^{2n}}{d\mu^{2n}}\exp(-\mu^2)$$

we see that

$$c_{2n}(t) = (-1)^n \nu t M^n H_{2n}(0).$$

The values $H_{2n}(0)$ are well known:

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!!$$

where $(2n-1)!! = (2n-1)(2n-3)\ldots 3, 1$. Thus, we have obtained the result

$$c_{2n}(t) = (2n-1)!!(2M)^n \nu t \tag{38}$$

for $n = 2, 3, \ldots$.

It is interesting that all the cumulants c_{2n} of our non-Gaussian random variable x are proportional to the time t. Only in the limiting case when $M \to +0$, the random variable x becomes Gaussian. (That is, the Gaussian nature is defined by the property that all cumulants, except c_2, must vanish.)

5. The Markov property

The multiplicative process $\xi(u)$ (for which we have coined the designation Poisson-modified Wiener process) is Markovian. This property of the process $\xi(u)$ is rather obvious. If t_1 is any intermediate time ($0 < t_1 < t$), the Fourier component $p(k, t)$ of the probability density $P(x, t|x_0)$ (cf integral (34)) fulfils the identity

$$p(k, t) = p(k, t_1)p(k, t-t_1) \tag{39}$$

(cf expression (20)). By applying the convolution theorem, we can at once prove the Chapman–Kolmogorov identity

$$P(x, t|x_0) = \int_{-\infty}^{\infty} dx_1\, P(x, t-t_1|x_1)P(x_1, t_1|x_0) \tag{40}$$

which is the necessary attribute of any Markov process.

If we dissect the time interval $(0, t)$ into N equal subintervals of length $\varepsilon = t/N$ and apply $(N-1)$-times the Chapman–Kolmogorov identity, we obtain the product integral representation

$$P(x, t|x_0) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \ldots \int_{-\infty}^{\infty} dx_{N-1}\, P(x, \varepsilon|x_{N-1}) \times P(x_{N-1}, \varepsilon|x_{N-2}) \ldots P(x_2, \varepsilon|x_1)P(x_1, \varepsilon|x_0) \tag{41}$$

assuming that the short-time functions $P(.\,, \varepsilon|\,.)$ are given in their explicit form in advance. The short-time functions P can, of course, be given by the sum (25). In section 7, we shall show, however, that for a good definition of the short-time function P (i.e. for $\varepsilon \to +0$), it is sufficient to consider the first two terms in the sum (25).

6. Three forms of the evolution equation for the probability density

In this section we shall pay attention to the problem of formulating the equation describing the evolution of a density profile $\psi(x, t)$ provided that the initial profile $\psi(x, 0)$ is known. The function $P(x, t|x_0)$ which we have calculated in the previous sections plays the role of the Green function of the evolution equation for $\psi(x, t)$:

$$\psi(x, t) = \int_{-\infty}^{\infty} dx_0\, P(x, t|x_0)\psi(x_0, 0) \tag{42}$$

(note the linearity!). We may start from the equation

$$\frac{\partial p(k, t)}{\partial t} = -\{\nu[1-\exp(-k^2M)]+k^2D\}p(k, t) \tag{43}$$

(whose validity follows from expression (20)).

6.1. The cumulant expansion form

From equations (33) and (36) we know that

$$-\{\nu[1-\exp(-k^2M)]+k^2D\}t = \ln X(-ik, t)$$

so that

$$\partial p(k, t)/\partial t = [(1/t)\ln X(-ik, t)]\, p(k, t).$$

After developing $\ln X(-ik, t)$ into the McLaurin series around the point $k=0$, one obtains the equation

$$\frac{\partial p(k, t)}{\partial t} = \sum_{n=0}^{\infty} \frac{c_{2n}(t)}{t} \frac{(-k^2)^n}{(2n)!}\, p(k, t).$$

Hence, when replacing k by the operator $-i\partial/\partial x$, we obtain the equation

$$\frac{\partial P(x, t|x_0)}{\partial t} = \sum_{n=0}^{\infty} \frac{c_{2n}(t)}{t} \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} P(x, t|x_0) \tag{44}$$

where the cumulants $c_{2n}(t)$ are given by formulae (37) and (38). By inserting them into equation (44), we obtain the equation

$$\frac{\partial P(x, t|x_0)}{\partial t} = \nu\left[\exp\left(M\frac{\partial^2}{\partial x^2}\right) - 1\right]P(x, t|x_0) + D\frac{\partial^2 P(x, t|x_0)}{\partial x^2} \tag{45}$$

and, owing to the linearity of relation (42), also the equation

$$\frac{\partial \psi(x, t)}{\partial t} = \nu\left[\exp\left(M\frac{\partial^2}{\partial x^2}\right) - 1\right]\psi(x, t) + D\frac{\partial^2 \psi(x, t)}{\partial x^2}. \tag{46}$$

This evolution equation has to be solved with respect to the prescribed initial condition

$$\psi(x, 0) = \psi_0(x). \tag{47}$$

In our case when we consider the whole x-axis, it is only required that the solution $\psi(x, t)$, with all its x-derivatives, must tend to zero if $|x| \to \infty$. Then the solution is unique and reads

$$\psi(x, t) = \int_{-\infty}^{\infty} \mathrm{d}x\, P(x, t|x_0)\psi_0(x_0) \tag{48}$$

where $P(x, t|x_0)$ is given by expression (21).

Formally, equation (46) is a functional differential equation (an equation containing an infinite number of differential operators). Its formulation is not superfluous since equation (48) with sum (21) is not validated in cases when boundary conditions play a role (when the process runs on an x-semiaxis or in a finite region). In such cases, expression (21) is no more the fundamental solution of the problem (since $P(x, t|x_0)$ must be found with respect to the boundary conditions) but, notwithstanding this, equation (46) itself still holds. We will not discuss problems with boundary conditions in this paper (see, however, the indications in section 6.2).

It is seen directly that if $M \to +0$, then equation (46) becomes the well-known second-order parabolic partial differential equation describing classical diffusion.

6.2. The integrodifferential form

This form has been preferred by Laskin (1989) for $D = 0$, and can be straightforwardly derived if one applies the convolution theorem to equation (43). In this way, we have obtained the equation

$$\frac{\partial P(x, t|x_0)}{\partial t} = \nu\int_{-\infty}^{\infty} \mathrm{d}x'[\Gamma(x - x', 2M) - \delta(x - x')]P(x', t|x_0) + D\frac{\partial^2 P(x, t|x_0)}{\partial x^2} \tag{49}$$

where we have used equation (23) for the Gaussian Γ. Evidently,

$$\Gamma(x - x', 2M) \to \delta(x - x') \tag{50}$$

for $M \to +0$ and so again (of course) equation (49) becomes the classical diffusion equation.

If $D = 0$, equation (49) resembles a master equation. This is the case studied by Laskin; in his words, the equation has a 'Boltzmann structure'. Our fundamental solution to equation (49) (i.e. the series (21)) generalizes his equation (3.9) (Laskin 1989).

After multiplying equation (49) by $\psi(x_0, 0)$ and integrating over x_0, we obtain the evolution equation

$$\frac{\partial\psi(x,t)}{\partial t} = \nu \int_{-\infty}^{\infty} \mathrm{d}x'[\Gamma(x-x', 2M) - \delta(x-x')]\psi(x', t) + D\frac{\partial^2\psi(x,t)}{\partial x^2}. \tag{51}$$

If a boundary (or two boundaries) is (are) present, then in the integral term of equation (51), we may use the analytical continuation of $\psi(x', t)$ (in x') even outside of the region for which we seek the solution $\psi(x, t)$. The absence of other differential x-operators than ∂^2/x^2 indicates that there will be practically the same boundary conditions (usually two) as for the diffusion or heat-transport equation (Carlslaw and Jaeger 1959) or for the Wiener process (Ikeda and Watanabe 1981, Gikhman and Skorokhod 1977, Gardiner 1986).

6.3. The integral form

Until now we have used equation (42) in its role as a linear relation between the functions $P(x, t|x_0)$ and $\psi(\,, t)$. We may, however, use it as an integral equation for $\psi(x, t)$ with the kernel $P(x, t|x_0)$ known *a priori*. Therefore, we write for any time instant t_0, such that $t > t_0 > 0$, the equation

$$\psi(x, t) = \int_{-\infty}^{\infty} \mathrm{d}x_0\, P(x, t-t_0|x_0)\psi(x_0, t_0). \tag{52}$$

(Of course, if $\psi(x, 0)$ is known, this does not mean that $\psi(x, t_0)$ is also known in advance.)

We can prove the equivalence of equation (52) with equation (46) as follows. We take $\varepsilon > 0$ as an infinitesimal time increment and write

$$\psi(x, t+\varepsilon) = \int_{-\infty}^{\infty} \mathrm{d}\xi\, P(x, \varepsilon|x+\xi)\psi(x+\xi, t). \tag{53}$$

We take the Taylor developments

$$\psi(x, t+\varepsilon) = \psi(x, \varepsilon) + \frac{\partial\psi(x,t)}{\partial t}\varepsilon + \ldots$$

$$\psi(x+\xi, t) = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{\partial^m\psi(x,t)}{\partial x^m}\xi^m.$$

After setting the latter series into the RHS of equation (53) and carrying out, term by term, integration with respect to ξ, utilizing the well-known results for the Gauss-Laplace integral,

$$\int_{-\infty}^{\infty} \mathrm{d}\xi\, \xi^{2m-1} \exp(-\alpha_n\xi^2) = 0$$

$$\int_{-\infty}^{\infty} \mathrm{d}\xi\, \xi^{2m} \exp(-\alpha_n\xi^2) = \sqrt{\pi}\frac{(2m-1)!!}{2^m}\alpha_n^{-(m+1/2)}$$

$m = 1, 2, \ldots$, where, in our case,

$$\alpha_n \equiv \alpha_n(\varepsilon) = \frac{1}{4(D\varepsilon + nM)}$$

we obtain the RHS of equation (53) as a function of the variable ε. Paying heed to the smallness of ε, we may develop this function in ε around zero, retaining only the zero-order term and the term linear in ε. So, when comparing the coefficient at ε on both sides of equation (53), we obtain equation (46).

7. A simplified form for the functional integral representing the probability density

For small values $\varepsilon > 0$, it is sufficient to use the function

$$P(x'', \varepsilon | x') = \exp(-\nu\varepsilon)(\Gamma(x''-x', 2D\varepsilon) + \nu\varepsilon\Gamma(x''-x', 2M)) \tag{54}$$

instead of the full sum required by expression (21) for values of t which are not infinitesimal. For Gaussians it is easy to calculate the Fourier transform and so, according to formula (34), we can get the function

$$p(k, t) = \exp(-\nu\varepsilon)[\exp(-k^2 D\varepsilon) + \nu\varepsilon \exp(-k^2 M)].$$

In contrast to the multiple convolution represented by the RHS of equality (41), the Fourier transform for $P(x, t|x_0)$ (by the convolution theorem) is much simpler:

$$p(k, t) = \exp(-\nu t)\left[\exp\left(-\frac{k^2 Dt}{N}\right) + \frac{\nu t}{N}\exp(-k^2 M)\right]^N$$

for $N \to \infty$, $\varepsilon = t/N \to 0$. However,

$$\lim_{N\to 0}[\ ]^N = \exp(-k^2 Dt)\left[1 + \frac{\nu t}{N}\exp(-k^2 M)\right]^N$$
$$= \exp(-k^2 Dt)\exp[\nu t \exp(-k^2 M)].$$

Thus, we have duly verified the t-dependence, as derived above for $p(k, t)$ (expression (20)). For $M \to +0$, the Gaussian $\Gamma(x''-x', 2M)$ in equation (54) is reduced to the delta function, $\delta(x''-x')$. Nevertheless, the component $p(k, t)$ does turn out to equal the simple exponential $\exp(-k^2 Dt)$, as has to be the case for classical Brownian motion.

8. The functional integral of the Feynman–Kac type

It is easy to include the rate function $V(x)$ (cf. section 2) in our equations:

$$\frac{\partial\psi(x, t)}{\partial t} = \nu\left[\exp\left(M\frac{\partial^2}{\partial x^2}\right) - 1\right]\psi(x, t) + D\frac{\partial^2\psi(x, t)}{\partial x^2} - V(x)\psi(x, t) \tag{55}$$

$$\frac{\partial\psi(x, t)}{\partial t} = \nu\int_{-\infty}^{\infty} dx'[\Gamma(x-x', 2M) - \delta(x-x')]\psi(x', t)$$
$$+ D\frac{\partial^2\psi(x, t)}{\partial x^2} - V(x)\psi(x, t). \tag{56}$$

The integral equation equivalent to equations (55) and (56) remains in the form of equation (52). We want only to indicate explicitly that the kernel is a functional of the function $V(x)$:

$$\psi(x, t) = \int_{-\infty}^{\infty} dx_0\, P_{\{V(\xi)\}}(x, t-t_0|x_0)\psi(x_0, t_0) \qquad (0 \leqslant t_0 \leqslant t) \tag{57}$$

$$P_{\{V(\xi)\}}(x, +0|x_0) = \delta(x-x_0). \tag{58}$$

By replacing, in formula (9), the free-diffusion function

$$G_0(x'', \varepsilon | x') = \Gamma(x''-x', 2D\varepsilon)$$

by the function $P(x'', \varepsilon | x')$ defined by expression (54) we can write down, in the N-approximation, the product integral

$$P_{\{V(\xi)\}}(x, t | x_0) = \exp(-\nu t)\left(\prod_{j=0}^{N-1} \int_{-\infty}^{\infty} d\xi_j (\Gamma(\xi_{j+1}-\xi_j, 2D\varepsilon) + \nu\varepsilon\Gamma(\xi_{j+1}-\xi_j, 2M)) \times \exp(-\varepsilon V(\xi_j))\right) \delta(\xi_0 - x_0) \tag{59}$$

with $\xi_N = x$, $\varepsilon = t/N$.

Apparently, expression (59) represents one of the simplest generalizations of the Feynman–Kac integral. To prove that we may use the product integral (59) in equation (57), we rewrite the latter in a form suited to the infinitesimal increment $\varepsilon > 0$:

$$\psi(x, t+\varepsilon) = \exp(-\nu\varepsilon)\exp(-V(x)\varepsilon) \int_{-\infty}^{\infty} d\xi (\Gamma(\xi, 2D\varepsilon) + \nu\varepsilon\Gamma(\xi, 2M))\psi(x+\xi, t). \tag{60}$$

As ε is small, we take

$$\exp(-\nu\varepsilon) = 1 - \nu\varepsilon \qquad \exp(-\varepsilon V(x)) = 1 - \varepsilon V(x).$$

When taking into account the Taylor development

$$\psi(x+\xi, t) = \sum_{n=1}^{\infty} (\xi^n/n!)\partial^n\psi(x, t)/\partial x^n$$

after performing the ξ-integration in $(-\infty, \infty)$ and using the formulae

$$\int d\xi\, \Gamma(\xi, \sigma^2) = 1 \qquad \int d\xi\, \xi^2\Gamma(\xi, \sigma^2) = \sigma^2$$

$$\int d\xi\, \xi^{2n-1}\Gamma(\xi, 2M) = 0 \qquad \int d\xi\, \xi^{2n}\Gamma(\xi, 2M) = (2M)^n(2n-1)!!$$

we can observe that equation (60) becomes the functional differential equation (55).

9. Suggestion of a quantum mechanical application: Quasiparticles with a non-parabolic dispersion law manifesting a point of inflexion

Let

$$E = E(k) \tag{61}$$

be a dispersion law for a certain kind of (non-interacting) quasiparticle (we have in mind a one-dimensional problem). We assume that $E(k)$ is defined on the whole k-axis as a real (even) analytical function, such that $E(k) \to \infty$ for $|k| \to \infty$. If the quasiparticles move in a field with a potential energy $V(x)$ $(-\infty < x < \infty)$, we may be interested in their canonical density matrix $C_\beta(x, x_0)$ (Feynman 1972). For the latter, we can write

down the Bloch equation (in the Wannier approach, say, as we consider a non-parabolic dispersion law (61) and then an effective mass of the quasiparticles):

$$-\frac{\partial C_\beta(x, x_0)}{\partial \beta} = \left[E\left(-\mathrm{i}\frac{\partial}{\partial x}\right) + V(x) \right] C_\beta(x, x_0) \tag{62}$$

$$C_{+0}(x, x_0) = \delta(x - x_0). \tag{63}$$

The canonical density matrix yields rich possibilities for quantum-theoretical calculations. Our intent here is to suggest that equation (62), with some adequate choice of the dispersion law (61), can become fully compatible with the theory of the Poisson-modified Wiener process dealt with in this paper. In equation (62), the thermodynamic parameter $\beta = 1/k_B T$ plays the same role as the time variable t in the previous sections.

If the dispersion law is parabolic,

$$E = E_p(k) = (\hbar^2/2m)k^2 \tag{64}$$

with the effective mass $m > 0$, then $C_\beta(x, x_0)$ can be represented by the Feynman-Kac integral. According to the theory described above, there is a further step for one special class of non-parabolic dispersion laws. We have in mind the function

$$E = E_{\text{non-p}}(k) = ak^2 + b[1 - \exp(-ck^2)] \tag{65}$$

with three positive constants—a, b and c. We can express $C_\beta(x, x_0)$ in the form of our generalized Feynman-Kac integral (product integral (59)). Then a is related to the diffusion coefficient D, b to the Poisson frequency ν and c to the parameter M.

The class of functions (65) is especially interesting for the following reason: with suitable combinations of the constants a, b and c, the curve $E = E_{\text{non-p}}(k)$ has two points of inflexion. The existence of points of inflexion is a feature of realistic dispersion laws. In particular, the function $E_{\text{non-p}}(k)$ may define the energy band of conduction electrons in a one-dimensional semiconductor. If we study a dynamic problem in which we can neglect the behaviour of the actual function $E(k)$ for high values of $|k|$ (for which $E_{\text{non-p}}(k)$ becomes unrealistic), we can try to fit the modelling function $E_{\text{non-p}}(k)$ with the known function $E(k)$ on an interval $0 < k < K$. The value K does not need to be as small as in the case of the parabolic modelling function $E_p(k)$ and a good fit can be reached. The fitting can be carried out, for instance, as follows. We fit the effective mass at $k_0 = 0$; this gives the equation

$$a + bc = \frac{1}{2}\frac{\partial^2 E(k)}{\partial k^2}\bigg|_{k=0} = \frac{\hbar^2}{2m_0}. \tag{66}$$

If k_i is the first value of k where the curve $E = E(k)$ exhibits the inflexion, we can require that k_i is also the position of the inflexion point on the curve $E = E_{\text{non-p}}(k)$ and equalize both the energies, E_p and $E_{\text{non-p}}$, at $k = k_i$. This gives the equations

$$ak_i^2 + b[1 - \exp(-ck_i^2)] = E(k_i) \tag{67}$$

$$a + bc(1 - 2ck_i^2)\exp(-ck_i^2) = 0. \tag{68}$$

(The value k_i results from the conditions: $\partial^2 E/\partial k^2 = 0$, $\partial E/\partial k > 0$. Equation (68) is the explicit transcription of the equation $\partial^2 E_{\text{non-p}}/\partial k^2 = 0$.)

It should be pointed out that there are also other reasonable possibilities to obtain the 'best' fit between $E(k)$ and $E_{\text{non-p}}(k)$. We refrain from a further discussion of this problem here.

10. Summary

In the present paper, we have developed a theory generalizing, in a sense, the concept of the Wiener process. Our modification of the Wiener process consists of the idea that the basic (background) Wiener process, characterized by a diffusion constant D, is accompanied by intercalatory Poisson random delta function pulses of equal strength M, applied stationarily (in the statistical sense) with some averaged frequency ν. Our process $\xi(u)$ (with u within an interval $(0, t)$) therefore becomes multiplicative. To describe this process we have paid much attention to some rather formal mathematical questions, such as the calculation of the conditional probability density $P(x, t|x_0)$ (where $\xi(0) = x_0$, $\xi(t) = x$), the calculation of all cumulants of the random variable x, the verification of the Chapman–Kolmogorov equality proving the Markov property of the process, and so on.

For a general density $\psi(x, t)$ (with a general initial density profile $\psi_0(x)$), we have derived three equivalent forms of its evolution equation. This equation has also been formulated with a term which (in two of the forms) corresponds to the potential energy term of the Schrödinger equation. We have succeeded in deriving a product integral for $P(x, t|x_0)$ which may be regarded as a generalization of the Feynman–Kac integral.

The theory of the present paper was inspired by one of two models which were put forward by Laskin (1989). If Laskin's model is discussed from the viewpoint of our theory, it corresponds to the limiting situation when the process $\xi(u)$ is formed solely by the Poisson random pulses, i.e. when the background Wiener process is absent. Certainly, the application proposed by Laskin—the dechanneling kinetics—is an attractive example where our theory could prove to be quite useful.

In section 9 of this paper, on the other hand, we have proposed another application of the Poisson-modified Wiener process $\xi(u)$: in a theory of quasiparticles (such as conduction electrons in crystals) whose dispersion law displays a point of inflexion. Although we have discussed the process $\xi(u)$ as one-dimensional, it is easy to employ its two- or three-dimensional version. This means that our application suggested in section 9 need not be relevant only for 'quantum wires' but also for crystalline thin films, as well as for bulk crystals.

Our last remark concerns the possibility which was suggested in section 3. Instead of the train of the Poisson random M-pulses and the background diffusion with the constant coefficient D, we may formulate another problem: when $D(t)$ is defined as the two-state random process with two diffusion coefficients, D and D_p, we may assume that not only the duration of the D-states but also the duration of the D_p states may be defined as a Poisson random variable. In other words, we can imagine random transitions from the diffusive D-state to the diffusive D_p state and vice versa. We will analyse this problem in another paper.

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